**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Calculus of variations

## Lecture 5. Lagrange problem for the functions family

We have the method of the analysis for the problem of minimization of the integral functional that depends from the unique unknown function. This problem can be transformed to the Euler equation. However, there exist many practical problems with functionals that depends from many unknown functions. We try to extend the previous results to these problems. We shell consider the movement of the body in the space as an example.

### 5.1. Problem statement

Consider the functional



where *F* is a known function, and unknown functions  …,  satisfy the boundary conditions

 (5.1)

 and  are given numbers, 

**Problem 5.1**. *Find the functions  that satisfy the boundary conditions* (5.1) *and minimize the functional I.*

Try to use the variational technique for solving this problem.

### 5.2. System of the Euler equations

Consider again the function of one variable



where *σ* is a number, and  is a smooth enough vector function that satisfy the homogeneous boundary conditions

 (5.2)

Then  which is called the *variation of the vector function u*, satisfy the boundary conditions (5.1). The functional *I* has the minimum at the point *u* if and only if the function *f* has the minimum at zero*.*

**Lemma 5.1**. *Suppose the function F has partial derivatives*  *and* ,  *Then there exists the derivative*

 (5.3)

**Proof**. Determine the value

Using Taylor formula, we have



where  as  Then



After dividing by *σ* and passing to the limit as  we get



Using the formula of the integration by parts and the equalities (5.2) we obtain



Then the previous equality is transformed to the formula (5.3).

The value at the right-hand side of the equality (4.3) is called the *variation of the functional* *I* at the point *u* and denotes by  Therefore, we have the equality

 (5.4)

**Theorem 5.1**. *If the smooth enough vector function u is a solution of the Problem* 5.1, *then it satisfies the system*

 (5.5)

**Proof**. Using Lemma 5.1, we transform the (5.4) to the equality



that is true for all functions  satisfying boundary conditions (5.2). We obtained before the analogical relation for the analysis of the scalar Lagrange problem (see Lecture 3). Then we used the Lagrange–Euler Lemma and determined the Euler equation. However, we had the scalar arbitrary function for the analogue of the previous equality.

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| **Question**: *How we can use the Lagrange–Euler Lemma now?* |

Fixe a concrete number *j*. Determine  if  and let  be arbitrary. Then we get



with arbitrary . Using Lagrange–Euler Lemma, we have



Therefore, the equalities (5.5) are true because the index *j* is arbitrary. €

Hence, Problem 5.1 is transformed to the analysis of the second order system of the ordinary differential equations (5.5) with boundary conditions

 (5.6)

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| **Conclusion**: *The boundary problem* (5.5), (5.6)  *can be used for the practical analysis of Problem* 5.1. |

Consider a practical application of this result.

### 5.3. Principle of least action

We considered before the fall of the body as an application of the Lagrange problem and the Euler equation. This is a partial case of the movement of the body, because, at first, this is the rectilinear movement; and secondly, this is the movement by the constant gravitational force only. Besides, the mass of the body was be constant. Now we consider the movement of the body with a variable mass  that moves in the space by the arbitrary variable force  This is the vector function with the components *F*1, *F*2 and *F*3. The movement of the body is described by the coordinates



The general characteristic of the considered mechanical system is the *action* that is the measure of the movement. This is the sum of the kinetic energy *K* and the work of the given force, namely



The kinetic energy determine by formula



Using formula

,

we find



The work is the scalar product of the force  and coordinate 



Then we find



We determined the action of the system at the concrete time *t.* Let us consider a time interval [*t*0,*t*1]. The completely action is the integral of *S*. Then we have



Suppose that the body is for the *t* = *t*1 and *t* = *t*2 in the points *х*0 = (*х*11, *х*12 , *х*13) and *х*2 = (*х*21 , *х*22 , *х*23). Therefore, we have the boundary conditions

*xi*(*t*1) = *x*1*i*, *xi*(*t*2) = *x*2*i*, *i* = 1,2,3.

By the *Principle of least action* the movement of the body from the point *х*0 to the point *х*1 is realize by the vector function  that realized the minimum of the action *I.* We have the Lagrange problem for the vector family. Using Theorem 5.1, we obtain the Euler equations (5.5). Determine the derivatives



Then we have the Euler equations



This is the system of the differential equations



We have the general form of the *Second Newton’s Law*. We determine that this result is the corollary of the Principle of least action.

### 5.4. First integral of the Euler equations

We considered the special form of the Lagrange problem for the scalar case. If we have the minimization problem for the functional



then its solution satisfies the first order differential equation



where *c* is an arbitrary constant.

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| **Question**: *What is an analogue of the last equality for the vector case?* |

Note that the second term at the left-hand side of the last equality is the scalar function *F.* Therefore, the analogue of this equality for the vector case is a unique equality. The first term there is the formal product of two vectors. We decide that this is the scalar product. Prove the following result.

**Lemma 5.1**.*Suppose the function F under the integral of Problem* 5.1 *does not depend from the independent variable x. Then the system of the Euler equations* (5.5) *can be transformed to the equality*

 (5.7)

*where c is an arbitrary constant*.

**Proof**. Find the value

.

Using the Euler equations, we obtain the equality



Therefore, the equality (5.7) is true.

**Corollary 5.1.** *The function*  *is the first integral**of the system*.

**Example**. Consider the previous physical example. We have the problem of minimization of the functional



Lemma 5.1 is applicable here, if the mass and the force do not depends from the time. Find the first integral

 (5.8)

Give the physical interpretation of this result. Consider the ***potential force*** that is the most important class of the physical forces. The force is potential is the work of this force is equal to zero for the movement along an arbitrary closed curve. For each potential force there exists a scalar characteristic *U* that is called the potential energy such that  We obtain the differential equations



After integration, we can find the potential energy by the equality



where *c* is an arbitrary constant. Put this result to the formula (5.8). We find



Using formula (5.7), we obtain the final result



Thus, the first integral of the considered system is the energy, and we ***determine the law of energy conservation.***

### Outcome

* The solution of Lagrange problem for the functions family satisfies the system of Euler equations with corresponding boundary conditions.
* The system of Euler equations is the necessary conditions of the local minimum of the given functional.
* The Principle of least action is the application of Lagrange problem for the functions family.
* Second Newton’s law is the corollary of the Principle of least action.

### Task 4. Lagrange problem with two unknown functions

Find the functions  which minimize the integral



with boundary conditions



The values of the parameters

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| variant |  |  |  |  |  |  |  |
| 1 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 2 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 3 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 4 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 |  | 0 | 2 | 0 | 2 | 0 | 1 |
| 6 |  | 0 | 1 | 0 | 1 | 1 | 0 |
| 7 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 8 |  | 0 | 1 | 1 | 0 | 1 | 0 |
| 9 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 10 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 11 |  | -1 | 0 | 0 | 1 | 0 | 1 |
| 12 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 13 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 14 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 15 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 16 |  | 0 | 1 | 0 | 1 | 1 | 0 |
| 17 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 18 |  | 0 | 1 | 0 | 2 | 0 | 1 |

Steps of the task.

1. Give the problem statement.
2. Determine the system of Euler equations.
3. Find the general solution of this system.
4. Find the solution of Euler equations, which satisfies the boundary conditions.
5. Show the graphs of these solutions.
6. Calculate the corresponding value of the given integral.

### References

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### Next step

We have the standard method for solving the problems of minimization for integral functionals with one or many unknown functions. However the functions under the integral depended only from the unknown functions and its first order derivatives. We will try to extend our results to the functional, which depends from the high derivatives of the unknown functions.